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THE ANISOTROPIC ELASTIC SOLID WITH AN ELLIPTIC HOLE OR RIGID INCLUSION

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Abstract—The two-dimensional problem of an elliptic hole in a solid of general anisotropy subject to an arbitrarily prescribed traction on the hole surface is studied. Stroh's complex formalism is adopted here but real-form solutions are obtained for the displacement and the hoop stress around the hole. For an arbitrarily prescribed traction, the solutions are in the form of an infinite series. However, through the use of a conjugate function they can be expressed in closed form directly in terms of the applied traction. We also consider an elliptic rigid inclusion subject to a force and a torque. Again, real-form solutions are obtained for the interface stress, the hoop stress around the rigid inclusion and the rotation of the rigid inclusion. When there is no torque applied at the inclusion, the traction vector at the surface of the rigid inclusion is in the direction of the applied force and is a constant when the ellipse is a circle. This is an unexpected result since the material surrounding the rigid inclusion is of general anisotropy.

1. INTRODUCTION

The problem of determining the stress distribution in a solid due to the presence of a hole or an inclusion has been a mathematically interesting and challenging problem. It is also an important problem in applications. A brief account of the history of research on the subject was given by Hwu and Ting (1989). The problem is particularly difficult to solve when the material is anisotropic. Among several formulations for anisotropic elasticity, Stroh's formalism (Stroh, 1958, 1962) has been proved to be powerful and elegant in solving two-dimensional problems (Barnett and Lothe, 1973, 1974, 1975, 1985; Asaro *et al.*, 1973; Chadwick and Smith, 1977). Recent advances in the theory allow us to present certain aspects of the solutions in a real form (Kirchner and Lothe, 1987; Ting, 1986, 1988a, b; Chadwick, 1989; Hwu and Ting, 1990; Li and Ting, 1989; Qu and Li, 1991; Suo, 1990).

The problem of an elliptic inclusion in a solid of general anisotropy subject to a uniform loading at infinity was studied by Hwu and Ting (1989), in which the inclusion can be a void, a rigid inclusion or an anisotropic elastic material different from the matrix. For the elliptic inclusion, real-form solutions are obtained for the stress inside the inclusion and around the interface boundary on the matrix. For the elliptic hole and elliptic rigid inclusion, real-form solutions are obtained for the hoop stress around the hole and the rotation of the rigid inclusion.

The present paper studies an elliptic hole subject to an arbitrarily prescribed traction on its surface and an elliptic rigid inclusion subject to a concentrated force and a torque. New derivations are presented which enable us to obtain the solutions in a simpler form. In Section 2, Stroh's formalism for two-dimensional anisotropic elasticity is outlined. Of the several different notations found in the literature, we follow the notation employed in Ting (1986). Some fundamental solutions which are needed in the present problem are presented in Section 3, and real-form solutions of these fundamental solutions along the hole boundary are derived in Section 4. In Section 5, we consider the problem of an elliptic hole subject to an arbitrarily prescribed traction on the surface of the hole. For certain special tractions, the hoop stress vector around the hole and the displacement of the hole boundary have a simple real form. In general, however, the solutions are in the form of an infinite series. In Section 6 we introduce the conjugate function through which the solutions are obtained in closed form directly in terms of the prescribed traction. In the last section, the rigid inclusion subject to a force and a torque is studied. The rotation of the rigid inclusion is obtained in the form of a quotient and found to be dependent on the torque only, not on the concentrated force. We are able to prove that the denominator of the quotient, which also appeared in Hwu and Ting (1989), is non-zero thus assuring the existence of the solution. When the rigid inclusion is subject to the concentrated force only, we found the unexpected result that the traction vector at the surface of the rigid inclusion is in the direction of the applied force and is a constant when the ellipse becomes a circle.

2. THE STROH FORMALISM

In a fixed rectangular coordinate system x_i, $i = 1, 2, 3$, let u_i, σ_{ij}, ε_{ij} be, respectively, the displacement, stress and strain. The strain–displacement equations, the stress–strain laws and the equations of equilibrium are

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \tag{1}$$

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks}, \tag{2}$$

$$C_{ijks}u_{k,sj} = 0, \tag{3}$$

where repeated indices imply summation, a comma stands for differentiation and C_{ijks} are the elastic constants which are assumed to be fully symmetric and positive definite. Assuming that u_i, $i = 1, 2, 3$, depend on x_1 and x_2 only, the general solution to (3) can be written in matrix notation as

$$\mathbf{u} = \sum_{\alpha=1}^{6} \mathbf{a}_\alpha f_\alpha(z_\alpha), \quad z_\alpha = x_1 + p_\alpha x_2, \tag{4}$$

in which $f_1, f_2, \ldots$ are arbitrary functions of their argument and p_α and $\mathbf{a}_\alpha$ are the eigenvalues and eigenvectors of the following eigenrelation:

$$\{\mathbf{Q}+p(\mathbf{R}+\mathbf{R}^{\mathrm{T}})+p^2\mathbf{T}\}\mathbf{a} = \mathbf{0}. \tag{5}$$

In (5), superscript T stands for the transpose and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 real matrices given by

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{6}$$

Equation (5) is obtained when we substitute (4) into (3). We see that $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite if the strain energy is positive. Since p_α cannot be real if the strain energy is positive (Eshelby *et al.*, 1953), there are three pairs of complex conjugates for p_α. We let

$$p_{\alpha+3} = \bar{p}_\alpha, \quad \mathrm{Im}\,(p_\alpha) > 0, \quad \alpha = 1, 2, 3,$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part. We then have

$$\mathbf{a}_{\alpha+3} = \bar{\mathbf{a}}_\alpha, \quad \alpha = 1, 2, 3.$$

For the displacement $\mathbf{u}$ to be real, we let

$$f_{\alpha+3} = \bar{f}_\alpha, \quad \alpha = 1, 2, 3,$$

and (4) becomes

$$\mathbf{u} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{3} \mathbf{a}_\alpha f_\alpha(z_\alpha)\right\}, \tag{7}$$

in which Re stands for the real part.

Introducing the vector

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q}+p\mathbf{R})\mathbf{a}, \tag{8}$$

where the second equality comes from (5), the stresses σ_{ij} obtained by substituting (4) into (1) and (2) can be written as

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \tag{9}$$

where the vector $\boldsymbol{\phi}$ is the stress function

$$\boldsymbol{\phi} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{3} \mathbf{b}_\alpha f_\alpha(z_\alpha)\right\}, \tag{10}$$

and $\mathbf{b}_\alpha$ is related to $\mathbf{a}_\alpha$ through (8). More generally, if $\mathbf{t}$ is the surface traction at a point on a curved boundary,

$$\mathbf{t} = \frac{\partial \boldsymbol{\phi}}{\partial s}, \tag{11}$$

where s is the arclength measured along the curved boundary in the direction such that, when one faces the direction of increasing s, the material is located on the right-hand side (Stroh, 1958). We see that (9) are special cases of (11) when the boundary is a plane parallel to the x_2-axis or the x_1-axis.

In many applications including the present one, f_1, f_2, f_3 have the same function form

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha), \quad \alpha \text{ not summed},$$

where q_α, $\alpha = 1, 2, 3$, are arbitrary complex constants. If we introduce the diagonal matrix

$$\langle f(z)\rangle = \mathrm{diag}\,[f(z_1), f(z_2), f(z_3)],$$

in which the angular brackets stand for the diagonal matrix, and the 3×3 complex matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3],$$

eqns (7) and (10) can be written as

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{A}\langle f(z)\rangle\mathbf{q}\}, \quad \boldsymbol{\phi} = 2\,\mathrm{Re}\,\{\mathbf{B}\langle f(z)\rangle\mathbf{q}\}, \tag{12}$$

$\mathbf{q}$ being the 3×1 matrix whose elements are q_1, q_2, q_3.

The two equations in (8) can be recast in the standard eigenrelation

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{13}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{14}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q}. \tag{15}$$

We see that $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric and $\mathbf{N}_2$ is positive definite. It can be shown (Ting, 1988c) that $-\mathbf{N}_3$ is positive semi-definite, and that $\mathbf{u}$ and $\boldsymbol{\phi}$ satisfy the differential equation (Chadwick and Smith, 1977)

$$\begin{bmatrix} \mathbf{u}_{,2} \\ \boldsymbol{\phi}_{,2} \end{bmatrix} = \mathbf{N} \begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix}. \tag{16}$$

Finally, the following three matrices introduced by Barnett and Lothe (1973)

$$\mathbf{H} = 2i\mathbf{A}\mathbf{A}^T, \quad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T, \quad \mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \tag{17}$$

$\mathbf{I}$ being the unit matrix, can be shown to be real. Moreover, $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite. The three matrices are related by

$$\mathbf{S}\mathbf{H} + \mathbf{H}\mathbf{S}^T = \mathbf{0}, \quad \mathbf{L}\mathbf{S} + \mathbf{S}^T\mathbf{L} = \mathbf{0}, \quad \mathbf{H}\mathbf{L} - \mathbf{S}\mathbf{S} = \mathbf{I}, \tag{18a}$$

which can be written as

$$\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} = -\mathbf{I}. \tag{18b}$$

Equations $(18a)_{1,2}$ imply that $\mathbf{SH}$ and $\mathbf{LS}$ are antisymmetric. It is readily shown that $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{S}\mathbf{L}^{-1}$ are also antisymmetric.

In the above presentation, we have tacitly assumed that the 6×6 matrix $\mathbf{N}$ is simple or semi-simple so that the six eigenvectors $\boldsymbol{\xi}$ span the six-dimensional space. Modifications required when $\mathbf{N}$ is non-semi-simple can be found in Chadwick and Smith (1977) and Ting and Hwu (1988). We hasten to add that the real-form solutions presented in this paper do not contain the eigenvalues p and the eigenvectors $\boldsymbol{\xi}$. Therefore, these solutions are valid for non-semi-simple $\mathbf{N}$.

The two-dimensional deformation presented here assumes that u_i, $i = 1, 2, 3$, are independent of x_3. This does not mean that u_3 vanishes although it does imply that $\varepsilon_{33} = 0$. The deformation is a generalization of plane strain of the isotropic elasticity. The in-plane displacements u_1 and u_2 are coupled with the anti-plane displacement u_3 due to the anisotropic property of the material. Therefore, u_3, ε_{31} and ε_{32} are in general non-zero.

3. FUNDAMENTAL SOLUTIONS

In an infinite anisotropic elastic material, let the boundary Γ of an elliptic hole or a rigid inclusion be given by

$$x_1(\psi) = a\cos\psi, \quad x_2(\psi) = b\sin\psi, \tag{19}$$

where a, b are the major and minor semi-axis, respectively, and ψ a real parameter; see Fig. 1. Consider the mapping

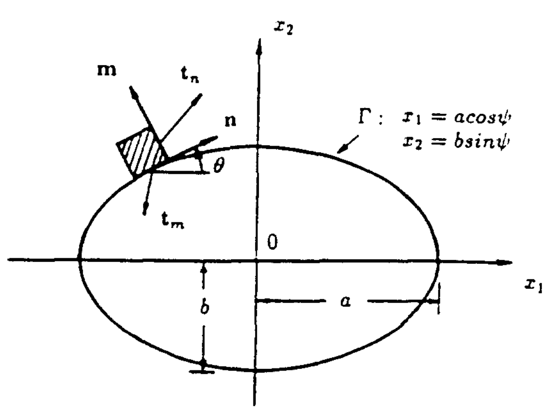

Fig. 1. Geometry of the elliptic hole or the elliptic rigid inclusion.

$$z_\alpha = c_\alpha \zeta_\alpha + d_\alpha \zeta_\alpha^{-1}, \quad \alpha \text{ not summed}, \tag{20}$$

where c_α, d_α are complex constants. Equation (20) transforms the complex variable z_α to a new complex variable ζ_α. When z_α is on the hole boundary Γ, let ζ_α be on a unit circle, i.e.

$$\zeta_\alpha|_\Gamma = e^{i\psi} = \cos\psi + i\sin\psi. \tag{21}$$

Substituting $(4)_2$ and (21) in (20) and using (19), we obtain

$$c_\alpha = \tfrac{1}{2}(a - ip_\alpha b),$$
$$d_\alpha = \tfrac{1}{2}(a + ip_\alpha b).$$

The roots of

$$\mathrm{d}z_\alpha/\mathrm{d}\zeta_\alpha = 0$$

are at

$$\zeta_\alpha^2 = \frac{d_\alpha}{c_\alpha} = \frac{a + ip_\alpha b}{a - ip_\alpha b}.$$

If p'_α, p''_α are, respectively, the real and imaginary parts of p_α, the absolute value of ζ_α is

$$|\zeta_\alpha| = \left\{\frac{(a - p''_\alpha b)^2 + (p'_\alpha b)^2}{(a + p''_\alpha b)^2 + (p'_\alpha b)^2}\right\}^{1/4} < 1$$

because a, b, p''_α are positive and non-zero. The roots are therefore located inside the unit circle and transformation (20) is one-to-one outside the hole with $\zeta_\alpha \to \infty$ as $z_\alpha \to \infty$.

One of the fundamental solutions for the elliptic hole is to choose

$$f(z_\alpha) = \ln \zeta_\alpha$$

in (12), where ζ_α is related to z_α through (20). As in Ting (1986, 1988a, b), we replace the complex constant $\mathbf{q}$ by

$$\mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{g}_0 + \mathbf{B}^{\mathrm{T}}\mathbf{h}_0,$$

where $\mathbf{g}_0$ and $\mathbf{h}_0$ are real constants. We then have the fundamental solution

$$\left.\begin{aligned} \mathbf{u}^{\mathrm{I}} &= 2\,\mathrm{Re}\,\{\mathbf{A}\langle \ln\zeta\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_0 + 2\,\mathrm{Re}\,\{\mathbf{A}\langle \ln\zeta\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_0, \\ \boldsymbol{\phi}^{\mathrm{I}} &= 2\,\mathrm{Re}\,\{\mathbf{B}\langle \ln\zeta\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_0 + 2\,\mathrm{Re}\,\{\mathbf{B}\langle \ln\zeta\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_0. \end{aligned}\right\} \tag{22}$$

Since $\ln \zeta_\alpha$ is a multi-valued function, we introduce a cut along $\psi = 0$. Although both $\mathbf{u}^{\mathrm{I}}$, $\boldsymbol{\phi}^{\mathrm{I}}$ become infinite as z_α goes to infinity, the stresses obtained from (9) vanish at infinity.

Another fundamental solution for the hole is to choose

$$\langle f(z)\rangle\mathbf{q} = \langle \zeta^{-k}\rangle(\mathbf{A}^{\mathrm{T}}\mathbf{g}_k + \mathbf{B}^{\mathrm{T}}\mathbf{h}_k), \quad k \text{ not summed},$$

in (12). Superimposing the solutions for $k = 1$ to infinity, we have

$$\left.\begin{aligned}\mathbf{u}^{\mathrm{II}} &= 2\sum_{k=1}^{\infty} \operatorname{Re}\{\mathbf{A}\langle\zeta^{-k}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_k + 2\sum_{k=1}^{\infty} \operatorname{Re}\{\mathbf{A}\langle\zeta^{-k}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_k,\\ \boldsymbol{\phi}^{\mathrm{II}} &= 2\sum_{k=1}^{\infty} \operatorname{Re}\{\mathbf{B}\langle\zeta^{-k}\rangle\mathbf{A}^{\mathrm{T}}\}\mathbf{g}_k + 2\sum_{k=1}^{\infty} \operatorname{Re}\{\mathbf{B}\langle\zeta^{-k}\rangle\mathbf{B}^{\mathrm{T}}\}\mathbf{h}_k,\end{aligned}\right\} \tag{23}$$

where $\mathbf{g}_k$, $\mathbf{h}_k$, $k = 1, 2, \ldots$, are real constants. We see that both $\mathbf{u}^{\mathrm{II}}$ and $\boldsymbol{\phi}^{\mathrm{II}}$ approach zero as z_α approaches infinity.

Noticing that

$$(\ln \zeta_\alpha)|_\Gamma = \mathrm{i}\psi, \quad \zeta_\alpha^{-k}|_\Gamma = \mathrm{e}^{-\mathrm{i}k\psi}$$

and using (17), the values of the fundamental solutions $\mathbf{u}^{\mathrm{I}}$, $\boldsymbol{\phi}^{\mathrm{I}}$, $\mathbf{u}^{\mathrm{II}}$, $\boldsymbol{\phi}^{\mathrm{II}}$ at Γ denoted by subscript Γ are

$$\mathbf{u}_\Gamma^{\mathrm{I}} = \psi\hat{\mathbf{h}}_0, \quad \boldsymbol{\phi}_\Gamma^{\mathrm{I}} = \psi\hat{\mathbf{g}}_0, \tag{24}$$

$$\left.\begin{aligned}\mathbf{u}_\Gamma^{\mathrm{II}} &= \sum_{k=1}^{\infty} (\mathbf{h}_k \cos k\psi - \hat{\mathbf{h}}_k \sin k\psi),\\ \boldsymbol{\phi}_\Gamma^{\mathrm{II}} &= \sum_{k=1}^{\infty} (\mathbf{g}_k \cos k\psi - \hat{\mathbf{g}}_k \sin k\psi).\end{aligned}\right\} \tag{25}$$

In the above, $\hat{\mathbf{g}}_k$, $\hat{\mathbf{h}}_k$, $k = 0, 1, 2, \ldots$, are related to $\mathbf{g}_k$, $\mathbf{h}_k$, by

$$\hat{\mathbf{h}}_k = \mathbf{S}\mathbf{h}_k + \mathbf{H}\mathbf{g}_k, \quad \hat{\mathbf{g}}_k = -\mathbf{L}\mathbf{h}_k + \mathbf{S}^{\mathrm{T}}\mathbf{g}_k, \tag{26}$$

or

$$\begin{bmatrix}\hat{\mathbf{h}}_k\\ \hat{\mathbf{g}}_k\end{bmatrix} = \begin{bmatrix}\mathbf{S} & \mathbf{H}\\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{h}_k\\ \mathbf{g}_k\end{bmatrix}.$$

It follows from (18b) that

$$\mathbf{h}_k = -(\mathbf{S}\hat{\mathbf{h}}_k + \mathbf{H}\hat{\mathbf{g}}_k), \quad \mathbf{g}_k = -(-\mathbf{L}\hat{\mathbf{h}}_k + \mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_k). \tag{27}$$

Equations (26), (27) allow us to determine $\hat{\mathbf{g}}_k$, $\hat{\mathbf{h}}_k$ in terms of $\mathbf{g}_k$, $\mathbf{h}_k$ and vice versa. If $\mathbf{g}_k$ and $\hat{\mathbf{g}}_k$ are known, we obtain from (26) and (18a)

$$\mathbf{h}_k = \mathbf{L}^{-1}(\mathbf{S}^{\mathrm{T}}\mathbf{g}_k - \hat{\mathbf{g}}_k), \quad \hat{\mathbf{h}}_k = \mathbf{L}^{-1}(\mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_k + \mathbf{g}_k). \tag{28}$$

If $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ are known, (27) and (18a) give

$$\mathbf{g}_k = -\mathbf{H}^{-1}(\mathbf{S}\mathbf{h}_k - \hat{\mathbf{h}}_k), \quad \hat{\mathbf{g}}_k = -\mathbf{H}^{-1}(\mathbf{S}\hat{\mathbf{h}}_k + \mathbf{h}_k). \tag{29}$$

Thus, if we can determine any two of the four constants $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, $\hat{\mathbf{h}}_k$, the remaining two are provided by (26), (27), (28) or (29).

4. STRESS ALONG THE HOLE BOUNDARY

Before we present solutions to the problems associated with an elliptic hole, we derive an explicit real-form expression for the stress along the elliptic hole boundary.

Let $\rho\, \mathrm{d}\psi$ be the infinitesimal arclength of the hole boundary Γ where

$$\rho(\psi) = (a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{1/2}.$$

The unit vectors tangential and normal to Γ as shown in Fig. 1 are

$$\mathbf{n}^T(\theta) = (\cos\theta, \sin\theta, 0),$$
$$\mathbf{m}^T(\theta) = (-\sin\theta, \cos\theta, 0), \tag{30}$$

$$\cos\theta = (a\sin\psi)\rho^{-1}(\psi), \quad \sin\theta = -(b\cos\psi)\rho^{-1}(\psi). \tag{31}$$

When the hole is a circle, i.e. when $a = b$, $\rho(\psi) = a$ which is independent of ψ and $\psi = \theta + \pi/2$. Let $\mathbf{t}_m$ be the traction on the hole surface. If n is the arclength of Γ measured in the direction of $\mathbf{n}$, we have by (11),

$$\mathbf{t}_m = \frac{\partial}{\rho\,\partial\psi}\boldsymbol{\phi} = -\boldsymbol{\phi}_{,n} = -(\boldsymbol{\phi}_{,1}\cos\theta + \boldsymbol{\phi}_{,2}\sin\theta). \tag{32}$$

The sign for $\mathbf{t}_m$ employed here is opposite of that employed in Hwu and Ting (1989). Substituting $\boldsymbol{\phi}^{\text{I}}$, $\boldsymbol{\phi}^{\text{II}}$ from $(24)_2$, $(25)_2$ we have

$$\mathbf{t}_m^{\text{I}} = \rho^{-1}(\psi)\hat{\mathbf{g}}_0, \tag{33}$$

$$\mathbf{t}_m^{\text{II}} = -\rho^{-1}(\psi)\sum_{k=1}^{\infty} k(\mathbf{g}_k \sin k\psi + \hat{\mathbf{g}}_k \cos k\psi). \tag{34}$$

Likewise, let $\mathbf{t}_n$ be the traction on the surface perpendicular to Γ; see Fig. 1. If m is the arclength measured along this surface in the direction of $\mathbf{m}$,

$$\mathbf{t}_n = -\boldsymbol{\phi}_{,m} = \boldsymbol{\phi}_{,1}\sin\theta - \boldsymbol{\phi}_{,2}\cos\theta, \tag{35}$$

which is the "hoop stress vector". The hoop stress t_{nn} and the two shear stresses t_{nm}, t_{n3} are

$$t_{nn} = \mathbf{t}_n\cdot\mathbf{n}, \quad t_{nm} = \mathbf{t}_n\cdot\mathbf{m}, \quad t_{n3} = \mathbf{t}_n\cdot\mathbf{e}_3, \tag{36}$$

where $\mathbf{e}_3$ is the unit vector in the x_3-direction, i.e.

$$\mathbf{e}_3^T = (0, 0, 1).$$

We will present an alternate formula for $(35)_1$ which avoids differentiation with m.

We generalize the matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ of (6) by

$$Q_{ik}(\theta) = C_{ijks}n_j(\theta)n_s(\theta),$$
$$R_{ik}(\theta) = C_{ijks}n_j(\theta)m_s(\theta),$$
$$T_{ik}(\theta) = C_{ijks}m_j(\theta)m_s(\theta),$$

in which $\mathbf{n}(\theta)$, $\mathbf{m}(\theta)$ are defined in (30). We see that $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$, $\mathbf{T}(\theta)$ reduce to $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ of (6) when $\theta = 0$. Let

$$\mathbf{N}(\theta) = \begin{bmatrix}\mathbf{N}_1(\theta) & \mathbf{N}_2(\theta)\\ \mathbf{N}_3(\theta) & \mathbf{N}_1^T(\theta)\end{bmatrix}, \quad \mathbf{N}_1(\theta) = -\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta),$$

$$\mathbf{N}_2(\theta) = \mathbf{T}^{-1}(\theta), \quad \mathbf{N}_3(\theta) = \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta) - \mathbf{Q}(\theta).$$

They reduce to $(14)_1$ and (15) when $\theta = 0$. It is shown in the Appendix that a generalization of (16) is

$$\begin{bmatrix} \mathbf{u}_{,m} \\ \boldsymbol{\phi}_{,m} \end{bmatrix} = \mathbf{N}(\theta) \begin{bmatrix} \mathbf{u}_{,n} \\ \boldsymbol{\phi}_{,n} \end{bmatrix}, \tag{37}$$

which converts differentiation in the direction **n** to the direction **m**. Hence,

$$\boldsymbol{\phi}_{,m} = \mathbf{N}_1^T(\theta)\boldsymbol{\phi}_{,n} + \mathbf{N}_3(\theta)\mathbf{u}_{,n}$$

and, along the hole boundary Γ,

$$\mathbf{t}_n = \mathbf{N}_1^T(\theta)\mathbf{t}_m - \mathbf{N}_3(\theta)\mathbf{u}_{\Gamma,n}. \tag{38}$$

Equation (38) applies to holes of general shape. Two special cases of (38) are worth emphasizing. Depending on whether the hole is a void or a rigid inclusion, we have

$$\mathbf{t}_n = -\mathbf{N}_3(\theta)\mathbf{u}_{\Gamma,n}, \quad \text{for a free surface} \tag{39}$$

$$\mathbf{t}_n = \mathbf{N}_1^T(\theta)\mathbf{t}_m, \quad \text{for a rigid inclusion.} \tag{40}$$

Equation (39) is obvious. As to (40), we observe that the displacement in the rigid inclusion may have a rigid body translation $\mathbf{u}_0$ and a rotation ω about the x_3-axis. Hence

$$\mathbf{u}_\Gamma = \mathbf{u}_0 + \omega\mathbf{e}_3 \times \mathbf{r}_\Gamma, \tag{41}$$

where $\mathbf{r}_\Gamma$ is the position vector of a point on Γ. Differentiating along the direction **n** yields

$$\mathbf{u}_{\Gamma,n} = \omega\mathbf{e}_3 \times \mathbf{n} = \omega\mathbf{m}. \tag{42}$$

With (42) and the identity (Ting, 1988b)

$$\mathbf{N}_3(\theta)\mathbf{m} = \mathbf{0},$$

(38) leads to (40).

For the elliptic hole under consideration, we write (38) as

$$\mathbf{t}_n = \mathbf{N}_1^T(\theta)\mathbf{t}_m + \mathbf{N}_3(\theta)\frac{\partial\mathbf{u}_\Gamma}{\rho\,\partial\psi}. \tag{43}$$

With $\mathbf{u}_\Gamma^I$, $\mathbf{u}_\Gamma^{II}$, $\mathbf{t}_m^I$, $\mathbf{t}_m^{II}$ presented in $(24)_1$, $(25)_1$, (33), (34), we have

$$\mathbf{t}_n^I = \rho^{-1}(\psi)\{\mathbf{N}_1^T(\theta)\hat{\mathbf{g}}_0 + \mathbf{N}_3(\theta)\hat{\mathbf{h}}_0\}, \tag{44}$$

$$\mathbf{t}_n^{II} = -\rho^{-1}(\psi)\mathbf{N}_1^T(\theta)\sum_{k=1}^{\infty} k(\mathbf{g}_k \sin k\psi + \hat{\mathbf{g}}_k \cos k\psi) - \rho^{-1}(\psi)\mathbf{N}_3(\theta)\sum_{k=1}^{\infty} k(\mathbf{h}_k \sin k\psi + \hat{\mathbf{h}}_k \cos k\psi). \tag{45}$$

5. THE HOLE SUBJECT TO PRESCRIBED TRACTIONS

Consider an elliptic hole which is subject to an arbitrarily prescribed traction $\boldsymbol{\tau}(\psi)$ on Γ while the stress at infinity vanishes. We let

$$\mathbf{u} = \mathbf{u}^{\mathrm{I}} + \mathbf{u}^{\mathrm{II}}, \quad \boldsymbol{\phi} = \boldsymbol{\phi}^{\mathrm{I}} + \boldsymbol{\phi}^{\mathrm{II}}. \tag{46a}$$

The right-hand sides are given in (22) and (23). Since the displacement must be single-valued, we see from $(24)_1$ that we must set

$$\hat{\mathbf{h}}_0 = \mathbf{0}. \tag{46b}$$

It follows from this and $(26)_1$ that $\mathbf{g}_0$ and $\mathbf{h}_0$ in (22) are related by

$$\mathbf{g}_0 = -\mathbf{H}^{-1}\mathbf{S}\mathbf{h}_0.$$

From $(24)_1$, $(25)_1$, (33) and (34), the displacement and the traction at Γ are

$$\mathbf{u}_\Gamma = \sum_{k=1}^{\infty} (\mathbf{h}_k \cos k\psi - \hat{\mathbf{h}}_k \sin k\psi), \tag{47}$$

$$\boldsymbol{\tau}(\psi) = \rho^{-1}(\psi)\hat{\mathbf{g}}_0 - \rho^{-1}(\psi) \sum_{k=1}^{\infty} k(\mathbf{g}_k \sin k\psi + \hat{\mathbf{g}}_k \cos k\psi). \tag{48}$$

Equation (48) leads to

$$\left.\begin{aligned} \hat{\mathbf{g}}_0 &= \frac{1}{2\pi}\int_0^{2\pi} \rho(\psi)\boldsymbol{\tau}(\psi)\,\mathrm{d}\psi, \\ \mathbf{g}_k &= -\frac{1}{k\pi}\int_0^{2\pi} \rho(\psi)\boldsymbol{\tau}(\psi)\sin k\psi\,\mathrm{d}\psi, \quad k \geqslant 1, \\ \hat{\mathbf{g}}_k &= -\frac{1}{k\pi}\int_0^{2\pi} \rho(\psi)\boldsymbol{\tau}(\psi)\cos k\psi\,\mathrm{d}\psi, \quad k \geqslant 1. \end{aligned}\right\} \tag{49}$$

We see that $2\pi\hat{\mathbf{g}}_0$ is the resultant force of $\boldsymbol{\tau}(\psi)$ applied at Γ. With $\mathbf{h}_k$, $\hat{\mathbf{h}}_k$ determined from (28), (47) can be written as

$$\mathbf{u}_\Gamma = -\mathbf{S}\mathbf{L}^{-1}\sum_{k=1}^{\infty}(\mathbf{g}_k \cos k\psi - \hat{\mathbf{g}}_k \sin k\psi) - \mathbf{L}^{-1}\sum_{k=1}^{\infty}(\mathbf{g}_k \sin k\psi + \hat{\mathbf{g}}_k \cos k\psi), \tag{50}$$

and the hoop stress vector from (43) is

$$\mathbf{t}_n = \mathbf{N}_1^{\mathrm{T}}(\theta)\boldsymbol{\tau}(\psi) + \mathbf{N}_3(\theta)\frac{\partial \mathbf{u}_\Gamma}{\rho\,\partial\psi}.$$

This can be rewritten as, using (48) and (50),

$$\mathbf{t}_n = \mathbf{G}_1(\theta)\boldsymbol{\tau}(\psi) + \rho^{-1}(\psi)\mathbf{G}_3(\theta)\left\{\mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_0 + \sum_{k=1}^{\infty} k(\mathbf{g}_k \cos k\psi - \hat{\mathbf{g}}_k \sin k\psi)\right\}, \tag{51}$$

where

$$\mathbf{G}_1(\theta) = \mathbf{N}_1^{\mathrm{T}}(\theta) - \mathbf{N}_3(\theta)\mathbf{S}\mathbf{L}^{-1}, \quad \mathbf{G}_3(\theta) = -\mathbf{N}_3(\theta)\mathbf{L}^{-1}. \tag{52}$$

It is clear that $\mathbf{G}_3(\theta)\mathbf{L}$ is a symmetric matrix, and so is $\mathbf{G}_1(\theta)\mathbf{L}$. The latter follows from the fact that (Kirchner and Lothe, 1986)

$$\begin{bmatrix} \mathbf{N}_1(\theta) & \mathbf{N}_2(\theta) \\ \mathbf{N}_3(\theta) & \mathbf{N}_1^T(\theta) \end{bmatrix}\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix}\begin{bmatrix} \mathbf{N}_1(\theta) & \mathbf{N}_2(\theta) \\ \mathbf{N}_3(\theta) & \mathbf{N}_1^T(\theta) \end{bmatrix}.$$

$\mathbf{G}_1(\theta)$ and $\mathbf{G}_3(\theta)$ share the following properties with $\mathbf{N}_1(\theta)$ and $\mathbf{N}_3(\theta)$. First, they are periodic in θ with periodicity π. Next, $\mathbf{G}_1^T(\theta)$ and $\mathbf{G}_3^T(\theta)$ satisfy the following identities which are valid when $\mathbf{G}_1^T(\theta)$ and $\mathbf{G}_3^T(\theta)$ are replaced, respectively, by $\mathbf{N}_1(\theta)$ and $\mathbf{N}_3(\theta)$ (Hwu and Ting, 1989):

$$\mathbf{G}_1^T(\theta)\mathbf{m}(\theta) = -\mathbf{n}(\theta), \quad \mathbf{G}_3^T(\theta)\mathbf{m}(\theta) = \mathbf{0}, \tag{53}$$

$$\left.\begin{aligned} \cos(\theta-\theta_0)\mathbf{G}_1^T(\theta)\mathbf{n}(\theta) &= \mathbf{G}_1^T(\theta)\mathbf{n}(\theta_0)-\sin(\theta-\theta_0)\mathbf{n}(\theta), \\ \sin(\theta-\theta_0)\mathbf{G}_1^T(\theta)\mathbf{n}(\theta) &= \mathbf{G}_1^T(\theta)\mathbf{m}(\theta_0)+\cos(\theta-\theta_0)\mathbf{n}(\theta), \\ \cos(\theta-\theta_0)\mathbf{G}_3^T(\theta)\mathbf{n}(\theta) &= \mathbf{G}_3^T(\theta)\mathbf{n}(\theta_0), \\ \sin(\theta-\theta_0)\mathbf{G}_3^T(\theta)\mathbf{n}(\theta) &= \mathbf{G}_3^T(\theta)\mathbf{m}(\theta_0). \end{aligned}\right\} \tag{54}$$

In (54), θ_0 is an arbitrary constant. Finally, like $\mathbf{N}_1(\theta)$, $\mathbf{G}_1(\theta)$ and $\mathbf{G}_3(\theta)$ are dimensionless. For isotropic materials, they have the expressions

$$\mathbf{G}_1(\theta) = \begin{bmatrix} \sin 2\theta & -\cos 2\theta & 0 \\ -\cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{G}_3(\theta) = \begin{bmatrix} 1+\cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & 1-\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the following, we consider three special tractions for $\boldsymbol{\tau}(\psi)$.

(i) For a uniform pressure p,

$$\boldsymbol{\tau} = p\mathbf{m}(\theta) = p\{-\mathbf{n}(0)\sin\theta+\mathbf{m}(0)\cos\theta\},$$

or, using (31),

$$\boldsymbol{\tau} = p\rho^{-1}(\psi)\{a\sin\psi\mathbf{m}(0)+b\cos\psi\mathbf{n}(0)\}.$$

Comparing this with (48), we have

$$\hat{\mathbf{g}}_0 = \mathbf{0}, \quad \mathbf{g}_1 = -pa\mathbf{m}(0), \quad \hat{\mathbf{g}}_1 = -pb\mathbf{n}(0), \quad \mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0}, \quad k > 1.$$

Equation (50) yields

$$\mathbf{u}_\Gamma = p\mathbf{S}\mathbf{L}^{-1}\begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + p\mathbf{L}^{-1}\begin{bmatrix} (b/a)x_1 \\ (a/b)x_2 \\ 0 \end{bmatrix}.$$

The hoop stress vector from (51) has the expression

$$\mathbf{t}_n = p\mathbf{G}_1(\theta)\mathbf{m}(\theta)+p\mathbf{G}_3(\theta)\begin{bmatrix} (b/a)\cos\theta \\ (a/b)\sin\theta \\ 0 \end{bmatrix}.$$

(ii) For a uniform in-plane shear stress τ, we let

$$\boldsymbol{\tau} = \tau\mathbf{n}(\theta) = \tau\{\mathbf{n}(0)\cos\theta+\mathbf{m}(0)\sin\theta\}.$$

Following the same procedure, we obtain

$$\mathbf{u}_\Gamma = \tau\mathbf{S}\mathbf{L}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \tau\mathbf{L}^{-1}\begin{bmatrix} (a/b)x_2 \\ -(b/a)x_1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{t}_n = \tau\mathbf{G}_1(\theta)\mathbf{m}(\theta) - \tau\mathbf{G}_3(\theta)\begin{bmatrix} -(a/b)\sin\theta \\ (b/a)\cos\theta \\ 0 \end{bmatrix}.$$

(iii) Consider the special case in which $\rho(\psi)\boldsymbol{\tau}(\psi)$ is a constant, i.e.

$$\rho(\psi)\boldsymbol{\tau}(\psi) = \mathbf{f}/2\pi, \tag{55}$$

where $\mathbf{f}$ is the total traction force. Equation (49) give us

$$\hat{\mathbf{g}}_0 = \frac{1}{2\pi}\mathbf{f},$$

$$\mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0}, \quad k \geqslant 1,$$

and (50), (51) reduce to

$$\mathbf{u}_\Gamma = \mathbf{0},$$

$$\mathbf{t}_n = \frac{1}{2\pi}\rho^{-1}(\psi)\{\mathbf{G}_1(\theta)+\mathbf{G}_3(\theta)\mathbf{S}^T\}\mathbf{f}.$$

The fact that $\mathbf{u}_\Gamma$ vanishes is not peculiar because we have ignored the rigid body translation and rotation of the entire body. What is peculiar here is that the displacement $\mathbf{u}_\Gamma$ is a constant, which means that the elliptic hole is not distorted. This means that if we fill the hole with a rigid inclusion and apply a concentrated force $\mathbf{f}$, the traction along the interface should be given by (55). We will see in Section 7 that this is indeed the case.

(iv) Consider the problem of an elliptic hole subject to a uniform stress σ_{ij}^∞ at infinity while the surface of the elliptic hole is traction free (Hwu and Ting, 1989). The solution to this problem can be separated into two parts. The first part is the uniform solution in which the stress is σ_{ij}^∞ everywhere. The second part is the "disturbed" state due to the presence of the hole. The solution to this part must satisfy the conditions that the stress vanishes at infinity while at the hole surface the traction τ_i is $\sigma_{ij}^\infty m_j(\theta)$. This is precisely the problem we are considering in this section. We have

$$\boldsymbol{\tau} = \boldsymbol{\sigma}^\infty\mathbf{m}(\theta),$$

where $\boldsymbol{\sigma}^\infty$ is the stress tensor σ_{ij}^∞. In particular, if $\sigma_{ij}^\infty = p\delta_{ij}$, $\boldsymbol{\tau} = p\mathbf{m}(\theta)$ which is the special case (i) studied earlier. For general σ_{ij}^∞, we follow the analysis of case (i) and find that $\mathbf{g}_k$, $\hat{\mathbf{g}}_k$ vanish for all k except

$$\mathbf{g}_1 = -a\mathbf{t}_2^\infty, \quad \hat{\mathbf{g}}_1 = -b\mathbf{t}_1^\infty,$$

in which

$$\mathbf{t}_1^x = \boldsymbol{\sigma}^x \mathbf{n}(0), \quad \mathbf{t}_2^x = \boldsymbol{\sigma}^x \mathbf{m}(0).$$

The displacement $\mathbf{u}_\Gamma$ and the hoop stress vector $\mathbf{t}_n$ at the hole boundary for the disturbed state are

$$\mathbf{u}_\Gamma = \mathbf{S}\mathbf{L}^{-1}(x_1\mathbf{t}_2^x - x_2\mathbf{t}_1^x) + \mathbf{L}^{-1}\left(\frac{b}{a}x_1\mathbf{t}_1^x + \frac{a}{b}x_2\mathbf{t}_2^x\right),$$

$$\mathbf{t}_n = \mathbf{G}_1(\theta)(\mathbf{t}_2^x \cos\theta - \mathbf{t}_1^x \sin\theta) + \mathbf{G}_3(\theta)\left(\frac{b}{a}\mathbf{t}_1^x \cos\theta + \frac{a}{b}\mathbf{t}_2^x \sin\theta\right).$$

The hoop stress t_{nn} for the disturbed state is, using $(36)_1$, (53) and (54),

$$t_{nn} = \mathbf{n}^T(0)\left\{\mathbf{G}_1(\theta)\mathbf{t}_2^x + \frac{b}{a}\mathbf{G}_3(\theta)\mathbf{t}_1^x\right\} - \mathbf{m}^T(0)\left\{\mathbf{G}_1(\theta)\mathbf{t}_1^x - \frac{a}{b}\mathbf{G}_3(\theta)\mathbf{t}_2^x\right\} - \mathbf{n}^T(\theta)\{\mathbf{t}_1^x \cos\theta + \mathbf{t}_2^x \sin\theta\}.$$

The last term

$$\mathbf{n}^T(\theta)\{\mathbf{t}_1^x \cos\theta + \mathbf{t}_2^x \sin\theta\}$$

is identical to the solution for the first part of the homogeneous solution. If we ignore this term, we obtain the total hoop stress which agrees with that derived in Hwu and Ting (1989).

6. THE CONJUGATE FUNCTION FOR ARBITRARY TRACTION $\tau(\psi)$

The special tractions considered at the end of last section are such that the series solutions (50) and (51) retain only one term. If the traction $\tau(\psi)$ is arbitrary, the series solutions in general retain infinite terms. One can avoid the infinite series if the conjugate function is employed.

Let a periodic function $f(\psi)$ be represented by

$$f(\psi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\psi + b_k \sin k\psi),$$

where a_0, a_k, b_k are constants. The conjugate function of $f(\psi)$, denoted by $[f(\psi)]^c$, is

$$[f(\psi)]^c = \sum_{k=1}^{\infty} (a_k \sin k\psi - b_k \cos k\psi).$$

It is shown by Bary (1964) that the conjugate function is expressible directly in terms of $f(\psi)$ as

$$[f(\psi)]^c = -\frac{1}{\pi}\int_0^{\pi} \frac{f(\psi+t) - f(\psi-t)}{2\tan(t/2)}\,\mathrm{d}t.$$

We see from (48) that the conjugate function of $\rho(\psi)\tau(\psi)$ is

$$[\rho(\psi)\tau(\psi)]^c = \sum_{k=1}^{\infty} k(\mathbf{g}_k \cos k\psi - \hat{\mathbf{g}}_k \sin k\psi) = -\frac{1}{\pi}\int_0^{\pi} \frac{\rho(\psi+t)\tau(\psi+t) - \rho(\psi-t)\tau(\psi-t)}{2\tan(t/2)}\,\mathrm{d}t.$$

The hoop stress vector $\mathbf{t}_n$ of (51) can therefore be written as

$$\mathbf{t}_n = \mathbf{G}_1(\theta)\boldsymbol{\tau}(\psi)+\rho^{-1}(\psi)\mathbf{G}_3(\theta)\{\mathbf{S}^T\hat{\mathbf{g}}_0+[\rho(\psi)\boldsymbol{\tau}(\psi)]^c\}.$$

If the hole is a circle, we have

$$\mathbf{t}_n = \mathbf{G}_1(\theta)\boldsymbol{\tau}(\psi)+\mathbf{G}_3(\theta)\left\{\frac{1}{a}\mathbf{S}^T\hat{\mathbf{g}}_0+[\boldsymbol{\tau}(\psi)]^c\right\},$$

where a is the radius of the circle.

The displacement $\mathbf{u}_\Gamma$ of (50) at the hole boundary can also be expressed in terms of $\boldsymbol{\tau}(\psi)$. If we differentiate (50) with ψ, we have

$$\begin{aligned}\frac{d}{d\psi}\mathbf{u}_\Gamma &= \mathbf{S}\mathbf{L}^{-1}\sum_{k=1}^{\infty}k(\mathbf{g}_k\sin k\psi+\hat{\mathbf{g}}_k\cos k\psi)-\mathbf{L}^{-1}\sum_{k=1}^{\infty}k(\mathbf{g}_k\cos k\psi-\hat{\mathbf{g}}_k\sin k\psi)\\ &= \mathbf{S}\mathbf{L}^{-1}\{\hat{\mathbf{g}}_0-\rho(\psi)\boldsymbol{\tau}(\psi)\}-\mathbf{L}^{-1}[\rho(\psi)\boldsymbol{\tau}(\psi)]^c.\end{aligned}$$

Hence

$$\mathbf{u}_\Gamma(\psi) = \mathbf{S}\mathbf{L}^{-1}\int_0^\psi\{\hat{\mathbf{g}}_0-\rho(t)\boldsymbol{\tau}(t)\}\,dt-\mathbf{L}^{-1}\int_0^\psi[\rho(t)\boldsymbol{\tau}(t)]^c\,dt+\mathbf{u}_\Gamma(0).$$

Unfortunately, $\mathbf{u}_\Gamma(0)$ has to be determined from (50). However, since it is a constant, and since the displacement is unique up to a rigid body translation and rotation, we may ignore $\mathbf{u}_\Gamma(0)$.

7. THE RIGID INCLUSION SUBJECT TO A FORCE AND A TORQUE

For the rigid inclusion subject to a resultant force $\mathbf{f}$ and a counter-clockwise torque T, we employ the same solution (46). From (24) and (25), the displacement and the stress function at the interface Γ are

$$\mathbf{u}_\Gamma = \sum_{k=1}^{\infty}(\mathbf{h}_k\cos k\psi-\hat{\mathbf{h}}_k\sin k\psi), \tag{56}$$

$$\boldsymbol{\phi}_\Gamma = \psi\hat{\mathbf{g}}_0+\sum_{k=1}^{\infty}(\mathbf{g}_k\cos k\psi-\hat{\mathbf{g}}_k\sin k\psi). \tag{57}$$

The equilibrium of the inclusion demands that

$$-\int_s \mathbf{t}_m\,ds+\mathbf{f} = \mathbf{0}.$$

Using (11) and (57), we have

$$\boldsymbol{\phi}_\Gamma(0)-\boldsymbol{\phi}_\Gamma(2\pi)+\mathbf{f} = \mathbf{0},$$

which yields

$$\hat{\mathbf{g}}_0 = \mathbf{f}/2\pi. \tag{58}$$

The rigid inclusion has no deformation but can have a rigid body translation (which can be taken to be zero) and a rigid body rotation given by

$$\mathbf{u}_\Gamma = \omega\{a\cos\psi\mathbf{m}(0) - b\sin\psi\mathbf{n}(0)\}, \tag{59}$$

where ω is the counter-clockwise rotation of the inclusion. Since the displacement at the interface Γ is continuous, (56) and (59) lead to

$$\mathbf{h}_1 = a\omega\mathbf{m}(0), \quad \hat{\mathbf{h}}_1 = b\omega\mathbf{n}(0), \quad \mathbf{h}_k = \hat{\mathbf{h}}_k = \mathbf{0}, \quad k > 1. \tag{60}$$

From (29), we have

$$\mathbf{g}_1 = \omega\mathbf{H}^{-1}\{b\mathbf{n}(0) - a\mathbf{S}\mathbf{m}(0)\}, \quad \hat{\mathbf{g}}_1 = -\omega\mathbf{H}^{-1}\{b\mathbf{S}\mathbf{n}(0) + a\mathbf{m}(0)\}, \quad \mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0}, \quad k > 1. \tag{61}$$

The traction $\mathbf{t}_m$ at the interface Γ is, from (33) and (34),

$$\mathbf{t}_m = \rho^{-1}(\psi)(\hat{\mathbf{g}}_0 - \hat{\mathbf{g}}_1\cos\psi - \mathbf{g}_1\sin\psi). \tag{62}$$

To determine the rotation ω, we use the condition that the total moment about the origin due to the traction $-\mathbf{t}_m$ and the torque T on the rigid inclusion vanishes. This means that

$$T - \int_0^{2\pi} \{x_1(\psi)\mathbf{m}(0) - x_2(\psi)\mathbf{n}(0)\}^{\mathrm{T}}\mathbf{t}_m\rho(\psi)\,\mathrm{d}\psi = 0. \tag{63}$$

Substituting (62) into (63) and using (61) we obtain

$$\omega = T/\pi U, \tag{64}$$

where

$$\begin{aligned} U &= b\mathbf{n}^{\mathrm{T}}(0)\mathbf{H}^{-1}\{b\mathbf{n}(0) - a\mathbf{S}\mathbf{m}(0)\} + a\mathbf{m}^{\mathrm{T}}(0)\mathbf{H}^{-1}\{a\mathbf{m}(0) + b\mathbf{S}\mathbf{n}(0)\} \\ &= b^2(\mathbf{H}^{-1})_{11} + a^2(\mathbf{H}^{-1})_{22} + 2ab(\mathbf{H}^{-1}\mathbf{S})_{21}. \end{aligned}$$

In the above, the subscripts outside the parentheses denote the components of the matrix. We see that the rotation ω depends on the torque T only, not on the resultant force $\mathbf{f}$.

The denominator U of ω can be shown to be positive and non-zero. Introducing the complex vector.

$$\mathbf{y} = \begin{bmatrix} -ib \\ a \\ 0 \end{bmatrix},$$

U can be rewritten in the form

$$U = \mathbf{y}^{\mathrm{T}}(\mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S})\bar{\mathbf{y}}$$

which is positive and non-zero because $(\mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S})$, the impedance matrix (Chadwick and Smith, 1977; Barnett and Lothe, 1985), is a positive definite Hermitian. Therefore ω exists.

Equation (62) can be rewritten as

$$\mathbf{t}_m = \frac{1}{2\pi}\rho^{-1}(\psi)\mathbf{f} - \frac{T}{\pi U}\mathbf{H}^{-1}\left\{\frac{a}{b}\mathbf{m}(0)\sin\theta + \frac{b}{a}\mathbf{n}(0)\cos\theta - \mathbf{S}\mathbf{m}(\theta)\right\}. \tag{65}$$

We see that if $T = 0$, $\rho(\psi)\mathbf{t}_m$ is a constant, which agrees with the observation made in Section 5. For the circular inclusion for which $\rho(\psi) = a = b$, (65) is simplified to

$$\mathbf{t}_m = \frac{1}{2\pi a}\mathbf{f} - \frac{T}{\pi U}\mathbf{H}^{-1}\{\mathbf{n}(\theta) - \mathbf{S}\mathbf{m}(\theta)\}. \tag{66}$$

In particular, if $T = 0$, the traction vector $\mathbf{t}_m$ at the circular interface is in the direction of $\mathbf{f}$ and is a constant. This is a rather unexpected result since no material symmetry has been assumed.

The hoop stress vector $\mathbf{t}_n$ is obtained from (40) in which $\mathbf{t}_m$ is given in (65) for the elliptic inclusion and in (66) for the circular inclusion.

8. CONCLUDING REMARKS

The real-form solutions obtained here are in terms of the real matrices $\mathbf{N}_i(\theta)$, $i = 1, 2, 3$ and the Barnett–Lothe matrices $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$. The matrices $\mathbf{N}_i(\theta)$ can be expressed directly in terms of the elastic constants. The matrices $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$, however, require solving the eigenrelation (13). An alternate integral formalism (Barnett and Lothe, 1973) for $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ avoids solving the eigenrelation but, except for special anisotropic materials, the integration requires a numerical approximation. Progress has been made recently in this respect. Explicit expressions for $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ are now available for orthotropic materials (Dongye and Ting, 1989; Chadwick and Wilson, 1990), cubic materials (Chadwick and Wilson, 1990; Chadwick and Smith, 1982) and transversely isotropic materials in which the axis of symmetry is in the (x_1, x_2) plane or the (x_1, x_3) plane (Chadwick, 1989). Recently, Ting (1991) obtained explicit expressions for $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ for monoclinic materials for which the plane of symmetry is at $x_3 = 0$.

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APPENDIX

We will derive (37) from (16). From (32), which also applies to **u**, we have

$$\begin{bmatrix} \mathbf{u}_{,n} \\ \boldsymbol{\phi}_{,n} \end{bmatrix} = \cos\theta \begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix} + \sin\theta \begin{bmatrix} \mathbf{u}_{,2} \\ \boldsymbol{\phi}_{,2} \end{bmatrix}.$$

Hence, using (16),

$$\begin{bmatrix} \mathbf{u}_{,n} \\ \boldsymbol{\phi}_{,n} \end{bmatrix} = (\cos\theta\mathbf{I} + \sin\theta\mathbf{N}) \begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix}. \tag{A1}$$

Likewise, we obtain from (35),

$$\begin{bmatrix} \mathbf{u}_{,m} \\ \boldsymbol{\phi}_{,m} \end{bmatrix} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{N}) \begin{bmatrix} \mathbf{u}_{,1} \\ \boldsymbol{\phi}_{,1} \end{bmatrix}. \tag{A2}$$

It is shown in (4.2) of Ting (1989b) that

$$(\cos\theta\mathbf{I} + \sin\theta\mathbf{N})^{-1} = \{\cos\theta\mathbf{I} - \sin\theta\mathbf{N}(\theta)\}.$$

Therefore we deduce from (A1), (A2) the relation

$$\begin{bmatrix} \mathbf{u}_{,m} \\ \boldsymbol{\phi}_{,m} \end{bmatrix} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{N})\{\cos\theta\mathbf{I} - \sin\theta\mathbf{N}(\theta)\} \begin{bmatrix} \mathbf{u}_{,n} \\ \boldsymbol{\phi}_{,n} \end{bmatrix}.$$

This leads to (37) due to the identity

$$(-\sin\theta\mathbf{I} + \cos\theta\mathbf{N})\{\cos\theta\mathbf{I} - \sin\theta\mathbf{N}(\theta)\} = \mathbf{N}(\theta),$$

which is obtained by a specialization of the identity (3.5) in Hwu and Ting (1990).